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Mordell-Weil rank の高い曲線族を構成する Néron の方法について

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1 Introduction.

Let $f : S \to B$ be a fibration of curves of genus g on a surface S over a curve B. We assume that S, B and f are defined over the rational number field $\mathbf{Q}$. Then, for each rational point b on B such that $\Gamma_b = f^{-1}(b)$ is nonsingular, we can consider the Mordell-Weil rank r_b of Γ_b over $\mathbf{Q}$, which is by definition the rank of the group of rational points on the Jacobian variety of Γ_b.

Néron gave a method to construct such a fibration such that

(1) B has infinitely many rational points.

(2) For infinitely many rational points b on B, the rank r_b is "large".

He started from the plane $S_0 = \mathbf{P}^2$. Let Λ be a pencil of genus g defined over $\mathbf{Q}$ on S_0. Let $S_1 \to S_0$ be the blow-up of S_0 at the base points of Λ, so that we obtain a fibration $f_1 : S_1 \to \mathbf{P}^1$ of curves of genus g. We assume that this fibration has a section. Then we take some base change $f : S \to B$ of $S_1 \to \mathbf{P}^1$ by a surjective morphism $B \to \mathbf{P}^1$ of curves defined over $\mathbf{Q}$.

$$\begin{array}{ccccc} S_0 & \longleftarrow & S_1 & \longleftarrow & S \\ \downarrow{\scriptstyle \Phi_\Lambda} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & = & \mathbf{P}^1 & \longleftarrow & B \end{array}$$

Let Γ denote the generic fibre of f. Then Γ is a curve over $\mathbf{Q}(B)$. Hence we can consider the Mordell-Weil rank of Γ over $\mathbf{Q}(B)$, which we denote by r. Then, by the specialization theorem of Néron, Silverman, Tate (cf. [N1], [L], [Se]), we obtain that there exists a finite subset Σ in the set of all rational points $B(\mathbf{Q})$ on B such that $r_b \geq r$ if $b \in B(\mathbf{Q}) \setminus \Sigma$. Therefore it is enough to find Λ and $B \to \mathbf{P}^1$ such that Γ has large r.

Néron [N2] claimed that he can construct:

(I) ($g=1$) a pencil Λ of cubic curves with $r \geq 11$ (B is an elliptic curve)

(II) ($g \geq 2$) a pencil Λ of degree $g+2$ with $r \geq 3g+7$ (B is an elliptic curve), and gave an outline of the construction. But he did not publish the precise proof.

Néron's claim for (I) was reproved and made effective by Shioda [Sh1] in 1991 applying his theory of Mordell-Weil lattices (MWL). Our purpose of this paper is to verify Néron's claim (II) applying Shioda's theory of MWL for higher genus fibration developed in [Sh2] and [Sh3].

In §2, we examine Néron's original construction. This part was done with Shioda ([Sh-U]). It turns out that Néron's claim is not completely correct, and that his original method proves only the existence of families with $r \geq 3g+6$. After §3 we modify Néron's method and construct families of curves of genus $g \geq 3$ with $r \geq 3g+7$.

In the meantime Shioda [Sh4] succeeded in constructing families of curves of any genus $g \geq 2$ with $r \geq 4g+7$ over $\mathbf{P}^n$ by a completely different method.

2 Néron's original construction.

In what follows all varieties are defined over the complex number field $\mathbf{C}$ if otherwise not mentioned. For a variety defined over $\mathbf{Q}$, a rational point means a $\mathbf{Q}$-rational point.

Let g be an arbitrary integer greater than one. To construct families of curves of genus g, Néron claimed the following:

Claim 2.1 (Néron [N2]) (See Figure 1) *We can choose on $\mathbf{P}^2$* i) *three distinct lines L_1, L_2, L_3 defined over $\mathbf{Q}$ with a common point O,* ii) *g distinct rational points $P_1, \ldots, P_g$, none of which lie on any line L_k, and* iii) *three rational points R_k on L_k ($k=1,2,3$) different from O, satisfying the following conditions:*

(a) *There exists an irreducible curve C_∞ of degree $g+2$ defined over $\mathbf{Q}$ such that O is a g-ple point and $P_1, \ldots, P_g$ are double points of C_∞, and C_∞ is tangent to L_k at R_k for each k.*

(b) *There exists an irreducible curve C_0 of degree $g+2$ defined over $\mathbf{Q}$ such that O is a g-ple point of C_0, and the intersection point $C_0 \cap C_\infty$ consists of $O, P_1, \ldots, P_g$ and other $2g+4$ rational points $Q_1, \ldots, Q_{2g+4}$ which are distinct and different from O, P_i, R_k for every i and k, and $C_0 \cap L_k$ consists of two distinct rational points R_k', R_k'' which are different from O and R_k for each k.*

Let $\tilde{\mathbf{P}} \to \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at O. Then the projection of $\mathbf{P}^2$ from O induces a fibration $p : \tilde{\mathbf{P}} \to \mathbf{P}^1$, whose fibres correspond to the lines on $\mathbf{P}^2$ passing through O.

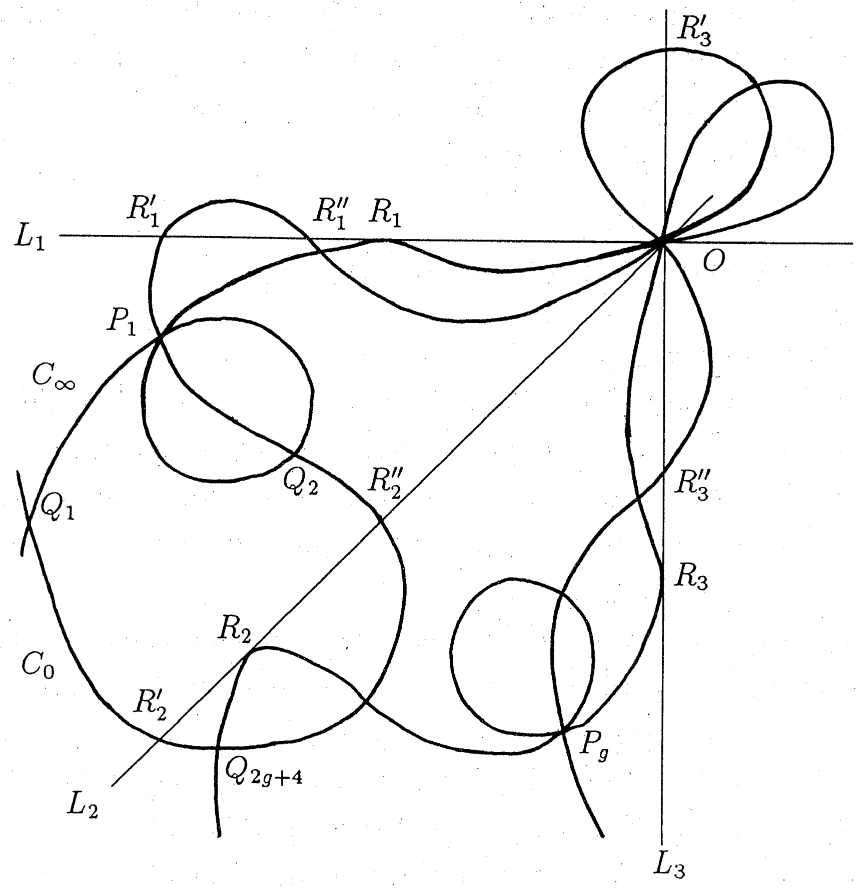


Figure 1: Curves on $\mathbf{P}^2$

It is clear that C_∞ is a rational curve. Hence the morphism p restricted to C_∞ defines a morphism of degree two between rational curves. Therefore there do not exist three fibres of p which are tangent to C_∞ at its smooth points. On the other hand, we can verify the statement in Claim 2.1 except the existence of the third line L_3. Let Λ denote the pencil on $\mathbf{P}^2$ spanned by C_∞ and C_0. Then we follow the diagram in §1. For the base change $S \to B$ we take the successive base changes of $S_1 \to \mathbf{P}^1$ by L_1 and L_2. Then we can show:

Theorem 2.2 *The rank of the Jacobian variety J of the curve Γ of genus g over $\mathbf{Q}(B)$ is at least $3g+6$.*

Moreover we can prove that the base curve B has infinitely many rational points. Hence we obtain:

Theorem 2.3 *There exists a non-empty open subset B_0 of $B(\mathbf{Q})$ such that $\{\Gamma_b\}_{b\in B_0}$ is an infinite family of curves of genus g over $\mathbf{Q}$ with rank at least $3g+6$.*

Remark. Néron claimed that one can construct a family of curves of genus g over $\mathbf{Q}$ with rank r at least $3g+7$ from the fibration $\mathbf{P}^2 - \to \mathbf{P}^1$ defined by Λ by changing the base three times with respect to the curves L_1, L_2 and L_3 in Claim 2.1. Hence the rank of curves we can construct via Néron's method is reduced by one.

For the detail of this section we refer to [Sh-U].

3 Construction of new pencils Λ.

In what follows we assume $g \geq 3$.

We take $S_0 = \mathbf{P}^1 \times \mathbf{P}^1$ in the diagram in §1 and let $\pi_i : S_0 \to \mathbf{P}^1$ be the projection to the i-th factor ($i = 1, 2$). Let F and G be a general fibre of π_1 and π_2 respectively. Then any complete linear system on S_0 is of the form $|mF + nG|$ for some m and n. We note $\dim |mF + nG| = mn + m + n$ if $m, n \geq 0$. For any point P on S_0, we denote by F_P the member in $|F|$ which passes through P.

Let Γ_1 be an irreducible curve in $|F+G|$ defined over $\mathbf{Q}$ and take two different rational points U and V on Γ_1. Then there exists an irreducible curve Γ_2 in $|2F+G|$ which is defined over $\mathbf{Q}$ and passes through U and V. The curves Γ_1 and Γ_2 meet also at the third point, say W. For the sake of simplicity, we take Γ_2 so that W is different from U and V. Take $F_1, \ldots, F_{g-3} \in |F|$ defined over $\mathbf{Q}$ such that $F_1, \ldots, F_{g-3}$, F_U, F_V and F_W are different from each other. Moreover take a general rational point Q_1 on Γ_2 and let G_0 denote the member in $|G|$ passing through Q_1, and R_k the intersection point $F_k \cap G_0$ ($1 \leq k \leq g-3$). We may assume $Q_1 \notin \Gamma_1$; $U, V \notin G_0$; and $R_1, \ldots, R_{g-3} \notin \Gamma_1, \Gamma_2$. Let U_1 [resp. V_1] denote the rational point $F_U \cap G_0$ [resp. $F_V \cap G_0$]. Let $P_1, \ldots, P_{g+2}$ [resp. $Q_2, \ldots, Q_{g+4}$] be general rational points on Γ_1 [resp. on Γ_2] such that no two points among $P_1, \ldots, P_{g+2}, Q_1, \ldots, Q_{g+4}, R_1, \ldots, R_{g-3}, U, V, W$ lie on a same fibre of π_1.

Set

$$\Lambda' = |(g+1)F + 2G - P_1 - \cdots - P_{g+2} - Q_1 - \cdots - Q_{g+4} - R_1 - \cdots - R_{g-3}|.$$

Since $\dim \Lambda' \geq 2$, there exists a curve C in Λ' which is defined over $\mathbf{Q}$ and passes through U_1 and V_1. Note that $CG = g+1$, $CF = 2$, $C\Gamma_1 = g+3$, $C\Gamma_2 = g+5$. Define rational points R_{g-2}, R'_k ($1 \leq k \leq g-2$), P_{g+3}, Q_{g+5}, U_2 and V_2 by the following equations as

0-cycles:

$$\begin{aligned}
CG_0 &= Q_1 + U_1 + V_1 + R_1 + \cdots + R_{g-3} + R_{g-2} \\
CF_k &= R_k + R'_k \quad (1 \leq k \leq g-2) \\
C\Gamma_1 &= P_1 + \cdots + P_{g+2} + P_{g+3} \\
C\Gamma_2 &= Q_1 + \cdots + Q_{g+4} + Q_{g+5} \\
CF_U &= U_1 + U_2 \\
CF_V &= V_1 + V_2,
\end{aligned}$$

where we set $F_{g-2} = F_{R_{g-2}}$. The configuration of the curves and the points defined above is shown in Figure 2.

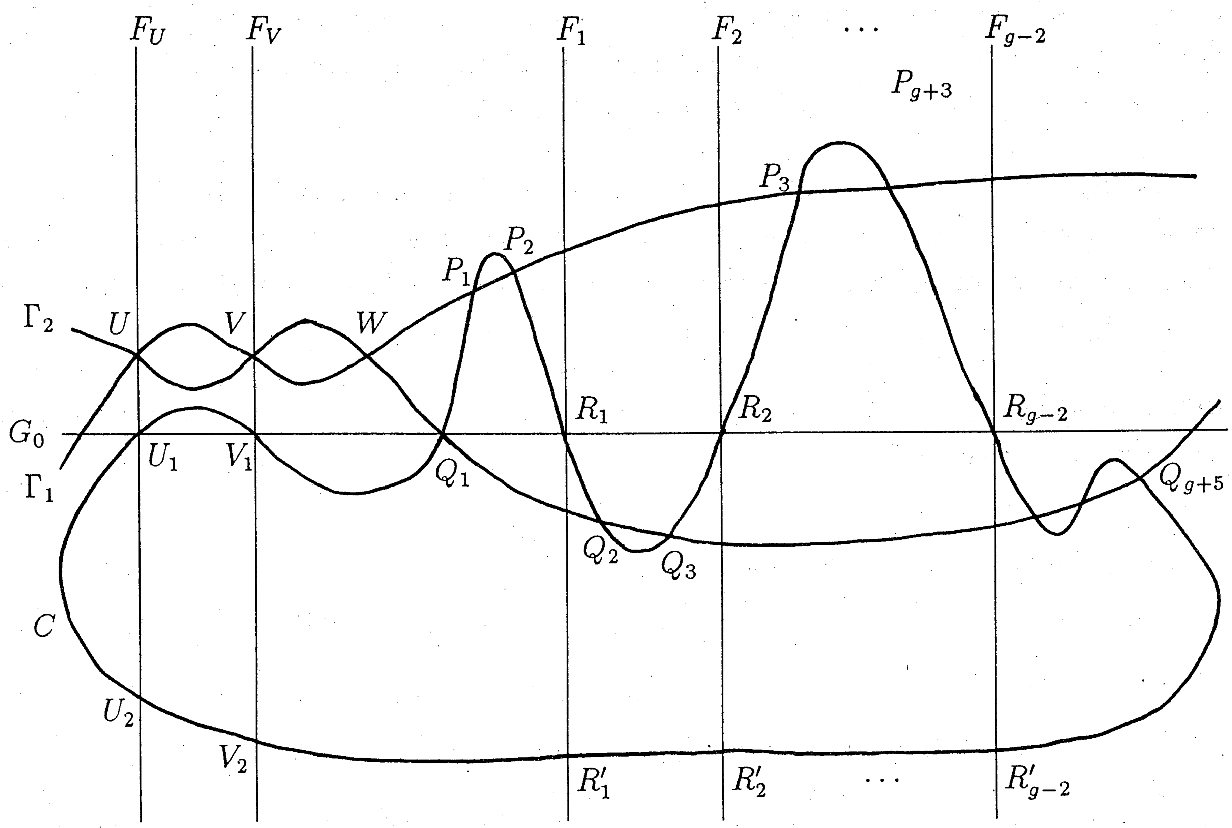


Figure 2: Curves on $S_0 = \mathbf{P}^1 \times \mathbf{P}^1$

Lemma 3.1 *If $P_1, \ldots, P_{g+2}$ and $Q_2, \ldots, Q_{g+4}$ are general, then C is irreducible, and hence the points R_{g-2}, R'_k, P_{g+3}, Q_{g+5}, U_2 and V_2 is well-defined. Moreover the following conditions are satisfied:*

(1) $R_k \neq R'_k (1 \leq k \leq g-2)$.

(2) $U_1 \neq U_2, V_1 \neq V_2$.

(3) *No two points among* $P_1, \ldots, P_{g+3}, Q_1, \ldots, Q_{g+5}, R_1, \ldots, R_{g-2}, U, V, W$ *lie on a same fibre of* π_1.

Proof. Set $\Lambda_1 = |(g+1)F + 2G - Q_1 - R_1 - \cdots - R_{g-3} - U_1 - V_1|$. Since Λ_1 contains $|F+2G| + F_{Q_1} + F_1 + \cdots + F_{g-3} + F_U + F_V$ and $|(g+1)F+G| + G_0$, we see that Λ_1 has no fixed component. Moreover we have $\dim \Lambda_1 \geq 2g+5$. Hence, if $P_1, \ldots, P_{g+2}$ on Γ_1 and $Q_2, \ldots, Q_{g+4}$ on Γ_2 are general, then we can find an irreducible member C in Λ_1 passing through these points.

Next, let us consider the restrictions of Λ_1 to G_0, Γ_1 and Γ_2. We have that $\Lambda_1|_{G_0} = Q_1 + R_1 + \cdots + R_{g-3} + U_1 + V_1 + |F||_{G_0}$, $\Lambda_1|_{\Gamma_1}$ is base point free, and that the base point of $\Lambda_1|_{\Gamma_2}$ is scheme-theoretically equal to the single point Q_1. Therefore, if $P_1, \ldots, P_{g+2}$, $Q_2, \ldots, Q_{g+4}$ are general so that C is general in Λ_1, then the conditions (1), (2), (3) are satisfied. Clearly C can be taken to be defined over $\mathbf{Q}$. $\square$

Assume that $P_1, \ldots, P_{g+2}, Q_2, \ldots, Q_{g+4}$ are general as in Lemma 3.1. Let $\Lambda = \{C_t\}_{t \in \mathbf{P}^1}$ be the subspace of Λ' spanned by $C_0 = C$ and $C_\infty = \Gamma_1 + \Gamma_2 + F_1 + \cdots + F_{g-2}$. We see that the base points of Λ is scheme-theoretically equal to $P_1 + \cdots + P_{g+3} + Q_1 + \cdots + Q_{g+5} + R_1 + \cdots + R_{g-2} + R'_1 + \cdots + R'_{g-2}$.

Lemma 3.2 *If* $P_1, \ldots, P_{g+2}$ *and* $Q_2, \ldots, Q_{g+4}$ *are general, then every member* C_t *in* Λ *with* $t \neq \infty$ *is irreducible.*

Proof. Let D be a reducible member in Λ.

Step 1. Suppose $D \geq \Gamma_1$: Set $D = \Gamma_1 + D_1$. Then D_1 is a member of

$$\begin{aligned} \Lambda_2 \quad &:= |gF + G - Q_1 - \cdots - Q_{g+5} - R_1 - \cdots - R_{g-2} - R'_1 - \cdots - R'_{g-2}| \\ &\subset |gF + G - Q_1 - \cdots - Q_{g+4} - R_1 - \cdots - R_{g-3}|. \end{aligned}$$

Since $\dim |gF + G - Q_1 - R_1 - \cdots - R_{g-3}| = g+3$, and since $Q_2, \ldots, Q_{g+4}$ are general with respect to $Q_1, R_1, \ldots, R_{g-3}$, we have either $\dim \Lambda_2 = 0$ or Γ_2 is a fixed component of Λ_2. Also in the former case Γ_2 is a fixed component of Λ_2, because $\Gamma_2 + F_1 + \cdots F_{g-2}$ always belongs to Λ_2. Hence we obtain $D = \Gamma_1 + \Gamma_2 + D_2$ with $D_2 \in |(g-2)F - R_1 - \cdots - R_{g-2} - R'_1 - \cdots - R'_{g-2}|$. Then it follows that $D_2 = F_1 + \cdots + F_{g-2}$, and so $D = C_\infty$.

Step 2. If $D \geq \Gamma_2$, then we obtain $D = C_\infty$ as in Step 1.

Step 3. Suppose $D = D_1 + D_2$ with $D_1 \in |mF + G - \sum_{i \in I} P_i - \sum_{j \in J} Q_j - \sum_{k \in K} R_k|$, $D_2 \in |(g+1-m)F + G - \sum_{i \notin I} P_i - \sum_{j \notin J} Q_j - \sum_{k \notin K} R_k|$ where $1 \leq m \leq g+1$, $I \subset \{1, \ldots, g+3\}$, $J \subset \{1, \ldots, g+5\}$, $K \subset \{1, \ldots, g-2\}$: We have $\dim |mF + G| +$

$\dim|(g+1-m)F+G| = 2g+4$. Since the $2g+5$ points $P_1, \ldots, P_{g+2}, Q_2, \ldots, Q_{g+4}$ are general on Γ_1 or Γ_2, we deduce that $D \geq \Gamma_1$ or Γ_2.

Step 4. Suppose $D = F' + D_1$ with $F' \in |F|$:

Case 1. $F' = F_{P_i}$ for some i $(1 \leq i \leq g+2)$: Then D_1 is a member of

$$\begin{aligned}\Lambda_3 &:= |gF + 2G - \sum_{\substack{1\leq l\leq g+3\\ l\neq i}} P_l - \sum_{1\leq j\leq g+5} Q_j - \sum_{1\leq k\leq g-2} R_k - \sum_{1\leq k\leq g-2} R'_k| \\ &\subset |gF + 2G - Q_1 - \sum_{1\leq k\leq g-3} R_k - \sum_{\substack{1\leq l\leq g+2\\ l\neq i}} P_l - \sum_{2\leq j\leq g+4} Q_j|.\end{aligned}$$

Since $\dim|gF+2G-Q_1-\sum_{1\leq k\leq g-3} R_k| = 2g+4$, it follows as in Step 1 that Λ_3 contains Γ_1 or Γ_2 as a fixed component.

Case 2. $F' = F_{Q_j}$ for some j $(2 \leq j \leq g+4)$: In this case we have $D_1 \in |gF + 2G - Q_1 - \sum_{1\leq k\leq g-3} R_k - \sum_{1\leq i\leq g+2} P_i - \sum_{\substack{2\leq l\leq g+4\\ l\neq j}} Q_l|$, and hence $D_1 \geq \Gamma_1$ or Γ_2 as in Case 1.

Case 3. $F' = F_k$ for some k $(1 \leq k \leq g-3)$ or $F' = F_{Q_1}$: We set $R_0 = Q_1$. Then we have $D_1 \in |gF + 2G - \sum_{\substack{0\leq l\leq g-3\\ l\neq k}} R_l - \sum_{1\leq i\leq g+2} P_i - \sum_{2\leq j\leq g+4} Q_j|$, and hence $D_1 \geq \Gamma_1$ or Γ_2.

Case 4. $F' \neq F_{P_i}(1 \leq i \leq g+2), F_{Q_j}(1 \leq j \leq g+4), F_k(1 \leq k \leq g-3)$: Then we have $D_1 \in |gF + 2G - Q_1 - \sum_{1\leq k\leq g-3} R_k - \sum_{1\leq i\leq g+2} P_i - \sum_{2\leq j\leq g+4} Q_j|$. However, if $P_1, \ldots, P_{g+2}, Q_2, \ldots, Q_{g+4}$ are general, then this linear system is empty since $\dim|gF + 2G - Q_1 - \sum_{1\leq k\leq g-3} R_k| = 2g+4$. □

4 Base change and configuration of the reducible fibres and sections.

Let Λ be the pencil defined in §1 such that Lemmas 3.1 and 3.2 hold.

Let $h_1 : S_1 \to S_0$ be the resolution of the base points of Λ and let $f_1 : S_1 \to \mathbf{P}^1$ be the morphism defined by Λ. Then S_1 is obtained by blowing up the $4g+4$ points P_i, Q_j, R_k, R'_k. For any divisor D on S_0, we denote its strict transform to S_1 also by the same letter D. Moreover we denote the exceptional curves of h_1 by the same letters as the corresponding points on S_0. They are sections of f_1. (See Figure 3.)

On S_1, the rational curves F_U and F_V are double sections of f_1. Let ι_U [resp. ι_V] denote the induced morphism F_U [resp. F_V] $\hookrightarrow S_1 \xrightarrow{f_1} \mathbf{P}^1$. Then ι_U [resp. ι_V] has two branch points. One of them is $t = \infty$. Let t_U [resp. t_V] denote the other branch point. Then t_U and t_V are rational points and $t_U, t_V \neq 0$. Let u be a coordinate of $F_U \cong \mathbf{P}^1$

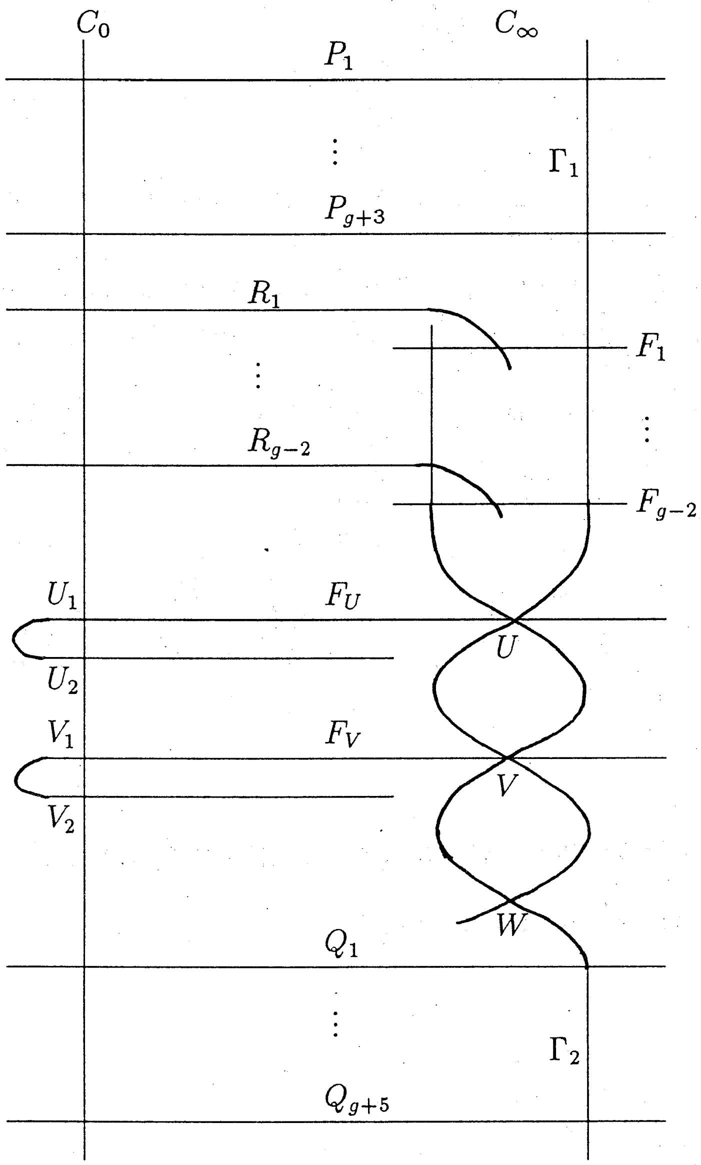


Figure 3: Curves on S_1

such that ι_U is defined by $u^2 = t - t_U$. Consider the base change $S_2' := S_1 \times_{\mathbf{P}^1} F_U \to F_U$ of $f_1 : S_1 \to \mathbf{P}^1$ by $\iota_U : F_U \to \mathbf{P}^1$. The induced morphism $S_2' \to S_1$ is of degree two and its branch locus is $D_\infty + D_{t_U}$. Hence S_2' has a singularity over each singular point of D_∞ and D_{t_U}. Let $S_2 \to S_2'$ be the minimal resolution of the singularities of S_2' and $f_2 : S_2 \to F_U, h_2 : S_2 \to S_1$ the induced morphisms. The fibration f_2 corresponds to the pencil $\{\tilde{D}_u\}_{u\in F_U}$ induced from the pencil $\{D_t\}_{t\in\mathbf{P}^1}$ on S_1 by h_2. Let F_U' and F_V' denote the strict transform of F_U and F_V by h_2 respectively. Then F_U' is a sum of two sections of f_2. If $t_U \neq t_V$, then F_V', is an irreducible rational curve with a node on $f_2^{-1}(\infty)$. Let B be the normalization of F_V' and let $S' := S_2 \times_{F_U} B \to B$ be the base change of $f_2 : S_2 \to F_U$ by $B \to F_V' \hookrightarrow S_2 \xrightarrow{f_2} F_U$. The singularity of S' corresponds to the singularity of the branch locus $h_2^{-1}(D_{t_V}) = \tilde{D}_{\sqrt{t_V - t_U}} + \tilde{D}_{-\sqrt{t_V-t_U}}$ of the induced morphism $S' \to S_2$. Let $S \to S'$ be the minimal resolution of the singularities of S' and $f : S \to B$, $h : S \to S_1$ the induced morphisms. If $t_U = t_V$, then we take $S = S_2$, $B = F_U$, $f = f_2$ and $h = h_2$. Then we obtained a new fibration $f : S \to B$ of curves of genus g. Note that B is a rational curve. On S, the strict transforms of both of F_U and F_V by h are sum of two sections of f, which we denote by $F_U^{(1)} + F_U^{(2)}$ and $F_V^{(1)} + F_V^{(2)}$ respectively. We denote the strict transforms of D_t, Γ_1, Γ_2, F_k, P_i, Q_j and R_k by h or h_2 by the same letters D_t, Γ_1, Γ_2, F_k, P_i, Q_j and R_k.

In order to calculate the height pairing in the next section, we need the configuration of all reducible fibres and sections P_i, Q_j, R_k, $F_U^{(1)}$ and $F_V^{(1)}$ of $f : S \to B$. From Lemma 3.2, the reducible fibres of f are fibres over D_∞ and possibly those over D_{t_U} and D_{t_V}. In this section we consider the reducible fibres on S_2, which are $\tilde{D}_\infty = f_2^{-1}(\infty)$ and possibly $\tilde{D}_0 = f_2^{-1}(0)$. We see (cf. Figure 3) that the fibre D_∞ on S_1 has $2g-1$ A_1-singularities. Let E_k and E_k' denote the exceptional curves on S_2 over the singular points $\Gamma_1 \cap F_k$ and $\Gamma_2 \cap F_k$ respectively of D_∞ $(1 \leq k \leq g-2)$ and we denote the exceptional curves over U, V and W by the same letters U, V and W. Then the configuration of curves near $\tilde{D}_\infty = \Gamma_1 + \Gamma_2 + \sum_{k=1}^{g-2}(F_k + E_k + E_k') + U + V + W$ is as in Figure 4.

Let T_U be the unique point of F_U on S_1 over t_U. Since $F_U D_{t_U} = 2$, T_U is at worst a double point of D_{t_U}. Hence the point on S_2' over T_U is an A_{n_U}-singularity for some n_U (we set $n_U = 0$ if C_{t_U} is non-singular at T_U and so S_2' is non-singular over T_U). Moreover, if D_{t_U} is singular at T_U, then a single blow-up at T_U separates the strict transforms of F_U and D_{t_U}. Set

$$n_U = \begin{cases} 2m_U & \text{if } n_U \text{ is even} \\ 2m_U - 1 & \text{if } n_U \text{ is odd.} \end{cases}$$

The exceptional set $h_2^{-1}(T_U)$ is a chain of n_U (-2)-curves, which we denote by $E_{U,1} + \cdots + E_{U,n_U}$. Then $F_U^{(1)}$ and $F_U^{(2)}$ meet only $E_{U,1}$ or E_{U,n_U}. We may assume that $F_U^{(1)} E_{U,1} = 1$

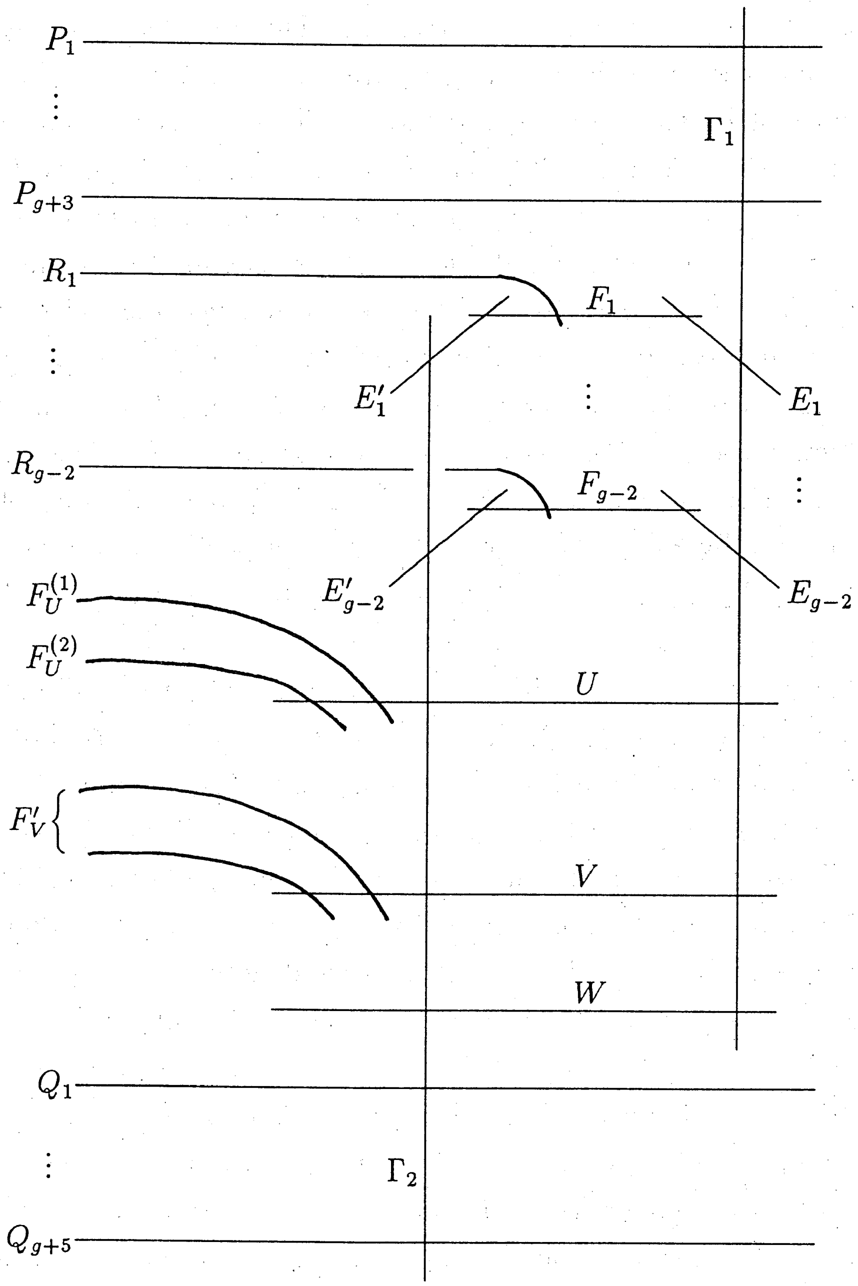


Figure 4: Curves on S_2 near $\tilde{C}_\infty$

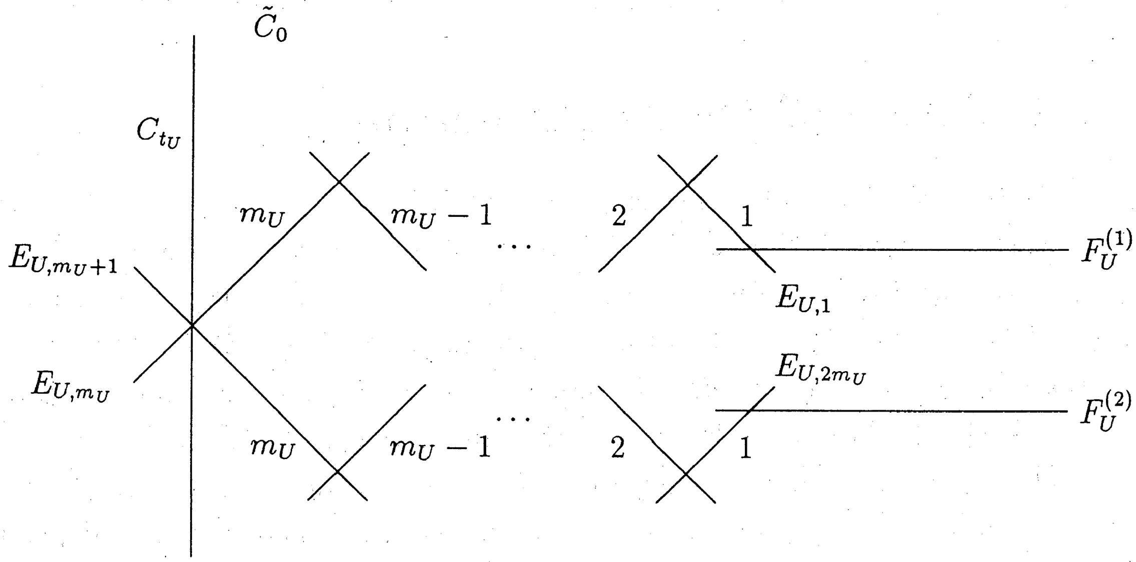
and $F_U^{(2)}E_{U,n_U} = 1$. The configuration of curves near $h_2^{-1}(T_U) \subset \tilde{D}_0$ is as in Figure 5, where the number attatched to each component $E_{U,l}$ is the multiplicity of $E_{U,l}$ in $\tilde{D}_0$.

5 Calculation of the height pairing.

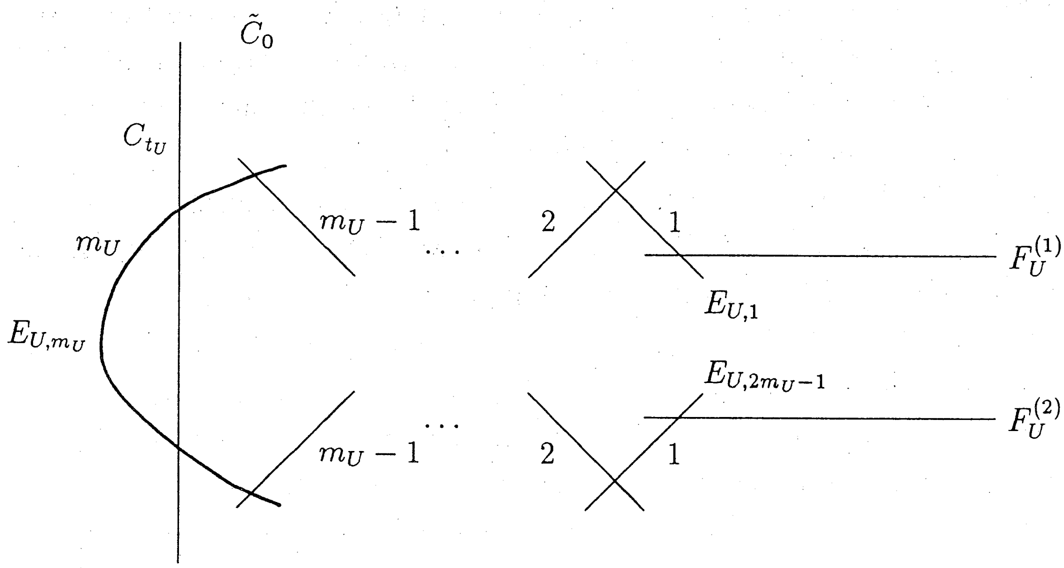
Let $f : S \to B$ be the fibration of genus g obtained in §2. This fibration is defined over $\mathbf{Q}$. Let Γ denote the generic fibre of f and J the Jacobian variety of Γ. Then Γ and J are defined over the function field $\mathbf{Q}(B)$ of B. The Mordell-Weil group M of J is defined as the group $J(\mathbf{Q}(B))/\tau Tr(\mathbf{Q})$ (modulo torsion), where (Tr, τ) is the $\mathbf{Q}(B)/\mathbf{Q}$-trace of J. The sections P_i, Q_j, R_k, $F_U^{(l)}$ and $F_V^{(l)}$ of f are defined over $\mathbf{Q}$ and hence can be regarded as points of $\Gamma(\mathbf{Q}(B))$. We may take Q_{g+5} as the zero-section of f and so the origin of J. Then P_i, Q_j, R_k, $F_U^{(l)}$, $F_V^{(l)}$ are also regarded as points of $J(\mathbf{Q}(B))$. We will show that the $3g+7$ points $P_1, \ldots, P_{g+3}$, $Q_1, \ldots, Q_{g+4}$, $R_1, \ldots, R_{g-2}$, $F_U^{(1)}$ and $F_V^{(1)}$ are independent in M by applying the theory of Mordell-Weil lattices for higher genus fibration developed in [Sh2] and [Sh3]. What we have to show is that the determinant of the matrix of the height pairing of these $3g+7$ points is not zero.

Recall that the height pairing is calculated as follows: Let T be the subgroup of the Néron-Severi group $NS(S)$ of S generated by the zero-section Q_{g+5}, a general fibre of f and all components of fibres of f which are disjoint from Q_{g+5}. For any section P of f, let $\varphi(P)$ denote the $\mathbf{Q}$-divisor on S such that (i) $\varphi(P) \equiv P$ (mod. $T \otimes \mathbf{Q}$) and (ii) $\varphi(P) \perp T$. Then for two sections P and Q of f, defined over $\mathbf{Q}$, the height pairing of the points in M corresponding to P and Q is equal to $-\varphi(P)\varphi(Q)$.

First we consider the case of $t_U = t_V$. Let us define T_V, n_V, m_V and $E_{V,1} + \cdots + E_{V,n_V}$ for V in the same way as T_U etc. for U. We may assume that $F_V^{(1)}E_{V,1} = 1$ and $F_V^{(2)}E_{V,n_V} = 1$. Let $\tilde{D}$ be a general fibre of f. Then the group T is generated by Q_{g+5}, $\tilde{D}$, Γ_1, $F_1, \ldots, F_{g-2}$, $E_1, E_1', \ldots, E_{g-2}, E_{g-2}'$, U, V, W, $E_{U,1}, \ldots, E_{U,n_U}$ and $E_{V,1}, \ldots, E_{V,n_V}$. (In fact $\tilde{D}_0$ has also components other than $C_{t_U} = C_{t_V}$, $E_{U,1}, \ldots, E_{U,n_U}$, $E_{V,1}, \ldots, E_{V,n_V}$ if D_{t_U} has singular points other than T_U and T_V. But the other singular points produce no effect on the calculation of the height pairing of the sections above. Hence we may assume that D_{t_U} is singular at worst only at T_U and T_V.) Since the morphism $S \to S_1$ is generically two to one, we have $P_i^2 = Q_j^2 = R_k^2 = -2$. Moreover we can calculate $F_U^2 = F_V^2 = -2$, $\Gamma_1^2 = \Gamma_2^2 = -(g+1)$, $F_k^2 = E_k^2 = E_k'^2 = -2$ and $D_{t_U}^2 = -(m_U + m_V)$. From these and Figures 4 and 5, we obtain the configuration of curves on S with their self-intersection numbers we need. As an example we show in Figure 6 one of the cases, in which n_U is even and n_V is odd.



(1) $n_U = 2m_U$



(2) $n_U = 2m_U - 1$

Figure 5: Curves on S_2 near $h_2^{-1}(T_U)$

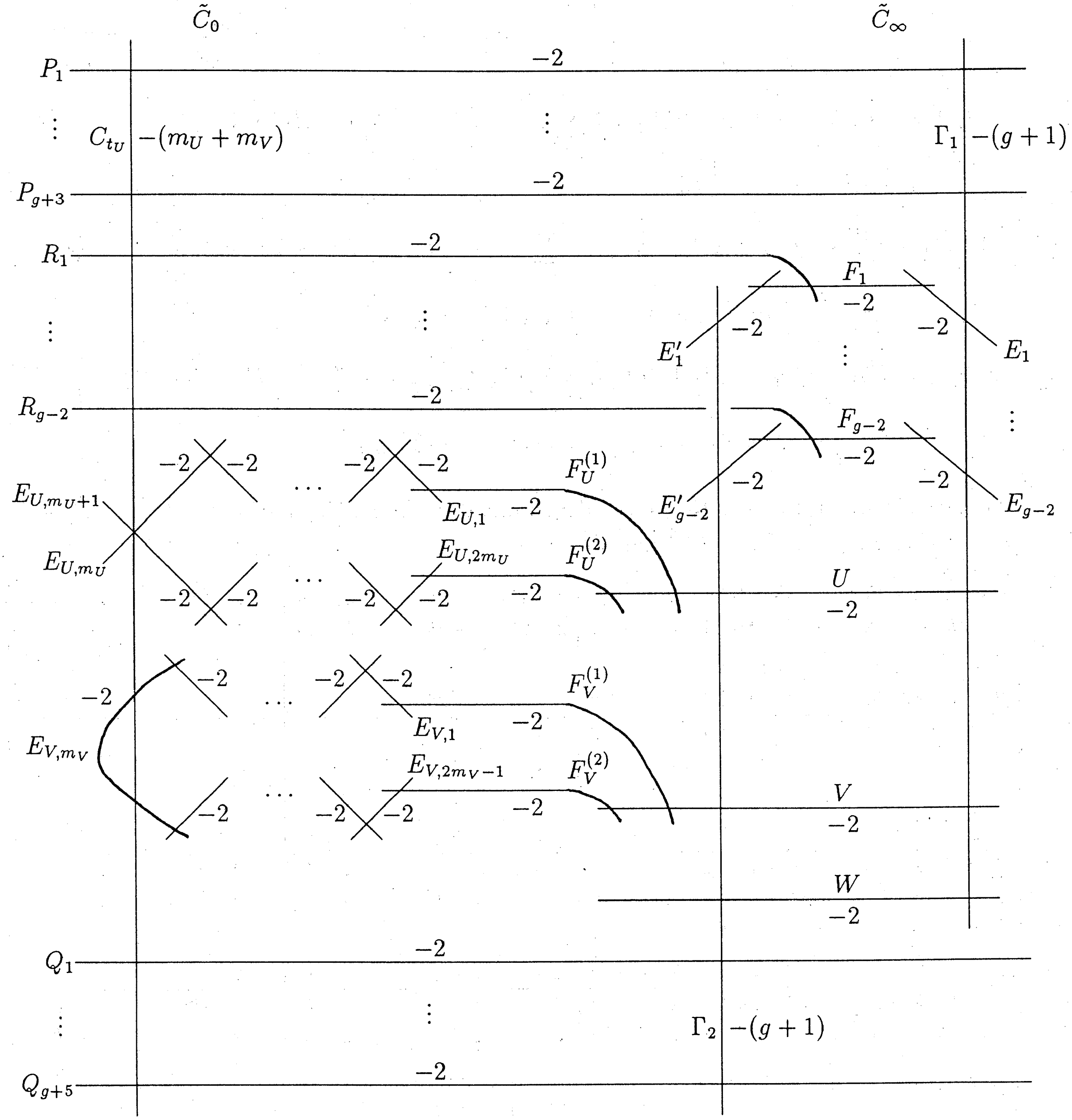


Figure 6: Sections and reducible fibres on S $(t_U = t_V, n_U = 2m_U, n_V = 2m_V - 1)$

In any case we can deduce:

$$
\begin{aligned}
\varphi(P_i) &= P_i - Q_{g+5} - 2\tilde{D} + 4h\Gamma_1 + h\sum_{p=1}^{g-2}(2F_p + 3E_p + E'_p) + 2h(U+V+W) \\
&\hspace{20em} (1 \le i \le g+3) \\
\varphi(Q_j) &= Q_j - Q_{g+5} - 2\tilde{D} \hspace{14em} (1 \le j \le g+4) \\
\varphi(R_k) &= R_k - Q_{g+5} - 2\tilde{D} + 2h\Gamma_1 + h\sum_{p=1}^{g-2}(F_p + \frac{3}{2}E_p + \frac{1}{2}E'_p) + h(U+V+W) \\
&\quad + F_k + \frac{1}{2}E_k + \frac{1}{2}E'_k \hspace{14em} (1 \le k \le g-2) \\
\varphi(F_U^{(1)}) &= F_U^{(1)} - Q_{g+5} - 2\tilde{D} + 2h\Gamma_1 + h\sum_{p=1}^{g-2}(F_p + \frac{3}{2}E_p + \frac{1}{2}E'_p) + h(U+V+W) + \frac{1}{2}U \\
&\quad + \frac{1}{n_U+1}(n_U E_{U,1} + (n_U - 1)E_{U,2} + \cdots + E_{U,n_U}) \\
\varphi(F_V^{(1)}) &= F_V^{(1)} - Q_{g+5} - 2\tilde{D} + 2h\Gamma_1 + h\sum_{p=1}^{g-2}(F_p + \frac{3}{2}E_p + \frac{1}{2}E'_p) + h(U+V+W) + \frac{1}{2}V \\
&\quad + \frac{1}{n_V+1}(n_V E_{V,1} + (n_V - 1)E_{V,2} + \cdots + E_{V,n_V}),
\end{aligned}
$$

where we set $h = 1/(g+4)$. Hence the determinant of the height pairing for $P_1, \dots, P_{g+3}$, $Q_1, \dots, Q_{g+4}$, $R_1, \dots, R_{g-2}$, $F_U^{(1)}$ and $F_V^{(1)}$ is:

$$
\left|\begin{array}{cccc|cccc|cccc|c|c}
4-4h & 2-4h & \cdots & 2-4h & 2 & 2 & \cdots & 2 & 2-2h & 2-2h & \cdots & 2-2h & 2-2h & 2-2h \\
2-4h & 4-4h & \cdots & 2-4h & 2 & 2 & \cdots & 2 & 2-2h & 2-2h & \cdots & 2-2h & 2-2h & 2-2h \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
2-4h & 2-4h & \cdots & 4-4h & 2 & 2 & \cdots & 2 & 2-2h & 2-2h & \cdots & 2-2h & 2-2h & 2-2h \\
\hline
2 & 2 & \cdots & 2 & 4 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\
2 & 2 & \cdots & 2 & 2 & 4 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 4 & 2 & 2 & \cdots & 2 & 2 & 2 \\
\hline
2-2h & 2-2h & \cdots & 2-2h & 2 & 2 & \cdots & 2 & 3-h & 2-h & \cdots & 2-h & 2-h & 2-h \\
2-2h & 2-2h & \cdots & 2-2h & 2 & 2 & \cdots & 2 & 2-h & 3-h & \cdots & 2-h & 2-h & 2-h \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2-2h & 2-2h & \cdots & 2-2h & 2 & 2 & \cdots & 2 & 2-h & 2-h & \cdots & 3-h & 2-h & 2-h \\
\hline
2-2h & 2-2h & \cdots & 2-2h & 2 & 2 & \cdots & 2 & 2-h & 2-h & \cdots & 2-h & \frac{7}{2}-h-\frac{n_U}{n_U+1} & 2-h \\
2-2h & 2-2h & \cdots & 2-2h & 2 & 2 & \cdots & 2 & 2-h & 2-h & \cdots & 2-h & 2-h & \frac{7}{2}-h-\frac{n_V}{n_V+1}
\end{array}\right|,
$$

which is equal to

$$
2^{2g+6}\frac{(3 - \frac{2n_U}{n_U+1})(3 - \frac{2n_V}{n_V+1})}{g+4} \neq 0.
$$

Next let us assume that $t_U \neq t_V$. Then there exist two fibres of $f : S \to B$ over each of the fibres $D_\infty, D_{t_U}, D_{t_V}$ of $f_1 : S_1 \to \mathbf{P}^1$. Moreover the morphism $h : S \to S_1$

is of degree 4, and hence all sections $P_1, \ldots, P_{g+3}$, $Q_1, \ldots, Q_{g+4}$, $R_1, \ldots, R_{g-2}$, $F_U^{(1)}$ and $F_V^{(1)}$ have self-intersection number -4. From these we see that the matrix of the height pairing of the sections above is obtained from that for the case $t_U = t_V$ by multiplying every entry by 2. Thus it follows that its determinant is:

$$2^{5g+13}\frac{(3-\frac{2n_U}{n_U+1})(3-\frac{2n_V}{n_V+1})}{g+4} \neq 0.$$

Therefore we have proved:

Theorem 5.1 *The rank of the Jacobian variety J of the curve Γ of genus g over $\mathbf{C}(B)$ is at least $3g+7$.*

6 Rational points on the base curve.

Let us prove that the base curve B of our fibration $f : S \to B$ defined in §2 has infinitely many rational points so as to show that f induces an infinite family of curves of genus g over $\mathbf{Q}$. From $\mathbf{Q}(F_U) = \mathbf{Q}(t)(u)$ and $\mathbf{Q}(F_V) = \mathbf{Q}(t)(v)$ where $u^2 = a(t-t_U)$ and $v^2 = b(t-t_V)$ for some $a, b \in \mathbf{Q}^\times$, we have

$$\mathbf{Q}(B) = \mathbf{Q}(u,v), \quad bu^2 - av^2 + ab(t_U - t_V) = 0.$$

Then what we need to show is that B has at least one rational point. Remember that on S_0, D_0 meets F_U [resp. F_V] at two rational points U_1 and U_2 [resp. V_1 and V_2]. Let $(0, u_1)$ and $(0, v_1)$ be the coordinates of U_1 and V_1 on S_1 respectively. Then we have $u_1^2 = -at_U$ and $v_1^2 = -bt_V$, and so $bu_1^2 - av_1^2 + ab(t_U - t_V) = 0$, hence we are done.

For any $b \in B$, let Γ_b denote the fibre of $f : S \to B$ over b. If b is a rational point, then Γ_b is a curve defined over $\mathbf{Q}$. Therefore, by the specialization theorem of Néron, Silverman, Tate (cf. [N1], [L], [Se]), we obtain the following:

Theorem 6.1 *There exists a non-empty open subset B_0 of $B(\mathbf{Q})$ such that $\{\Gamma_b\}_{b\in B_0}$ is an infinite family of curves of genus g over $\mathbf{Q}$ with rank at least $3g+7$.*

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